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INVESTIGATION OF AUTO-OSCILLATIONS OF A CONTINUOUS MEDIUM, OCCURRING AT LOSS OF STABILITY OF A STATIONARY MODE

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The theory developed in [1] is applied to a certain class of ordinary differential equations in a Hilbert space. Sufficient conditions are indicated for the appearance of auto-oscillations on the passage of a certain parameter through its critical value (existence of that value is established by the theory of normal operators). A nonlinear parabolic equation is investigated as an example.

The auto-oscillating modes are given for the general equation of [1] in the form of series in fractional powers of the supercritical parameter δ, and in the form of series in powers of the amplitude coefficient accompanying the neutral perturbation in the Fourier expansion. The perturbation theory is used to study the stability of the auto-oscillations. The terminology and the basic notation of [1] are all retained.

1. A class of equations with auto-oscillating solutions. Let us consider the ordinary differential equation

$$dv/dt + Av - \lambda Bv = K(v, \lambda) \tag{1.1}$$

in the Hilbert space H, under the following assumptions:

1) The operator A is self-conjugate, positive definite and coercive (maps D_A on all H in one-to-one correspondence). The operator A^{-1} is completely continuous.

2) The operator B is linear, with a domain of definition dense in H, and is subordinate to the operator $A^{1/2}$ in the sense that $D_{B_r} \supset D_{A^{1/2}}$, $D_{B_i} \supset D_{A^{1/2}}$ and the operators $A^{-1/2}B_r$ and $A^{-1/2}B_i$ are bounded (B_r and B_i denote the real and imaginary part of the operator B).

3) The real part B_r of B is not zero.

4) The operator B_i commutes with $A^{-1}B_r$ and the operator B_r commutes with A^{-1}

$$B_i(A^{-1}B_r)u = (A^{-1}B_r)B_iu$$
$$B_rA^{-1}u = A^{-1}B_ru \qquad (u \in D_{A^{1/2}}) \tag{1.2}$$

By W_p we denote the space of vector functions $v(t)$ whose values belong to H defined for $t \in [0, 2\pi]$, continuous and having the following finite norm

$$\|v\|_{W_p} = \left\{\int_0^{2\pi}\left[\left\|\frac{dv}{d\tau}\right\|_H^p + \|Av(\tau)\|_H^p\right]d\tau\right\}^{1/p} + \max_\tau \|v(\tau)\|_H$$

5) The nonlinear operator K acts continuously from W_2 into $L_2((0, 2\pi), H)$ and is analytic over the set (v, λ) near the point $(0, \lambda_0)$ of the space $H \times R$ for any λ_0. Moreover it is assumed that the Frechet derivative $K_v(0, \lambda) = 0$ for all λ.

We shall assume that an operation of complex conjugation is defined on the space $H: u \to u^*$ and the operators A, B and K are real: $(Au)^* = Au^*$ etc.

Let us consider the following eigenvalue problem

$$A\varphi_0 - \lambda B_r\varphi_0 = 0 \qquad (1.3)$$

Inverting the operator A we arrive at the equation

$$\varphi_0 = \lambda T\varphi_0, \qquad T = A^{-1}B_r \qquad (1.4)$$

The operator T is self-conjugate and completely continuous in the energy space H_1 of the operator A. Indeed, for any $u,\ v \in H_1$ we have

$$(Tu, v)_{H_1} = (A^{1/2}A^{-1}B_ru, A^{1/2}v)_H = (B_ru, v)_H = (u, B_rv)_H = (u, Tv)_{H_1} \qquad (1.5)$$

Conditions (1) and (3) imply that $T \neq 0$. By the Hilbert-Schmidt theorem there exists at least one nonzero real eigenvalue λ_0 of the operator T. Conditions (1) and (2) imply that any eigenvector φ of T satisfies (1.3) (in particular $\varphi \in D_A$). Condition (4) implies that a (finite-dimensional) characteristic subspace H_0 of the operator T corresponding to the eigenvector λ_0, is invariant with respect to the operator B_i.

Operator B_i is symmetric on H_0 Let us assume that it is nondegenerate on H_0 and has $\dim H_0$ eigenvectors. Let φ be one of these vectors. Then

$$-\lambda_0 B_i\varphi = \omega_0\varphi \qquad (1.6)$$

where ω_0 is a real number different from zero. Applying the operation of complex conjugation to the above equation we find that φ^* is also an eigenvector of B_i with a corresponding eigenvalue ω_0 / λ_0.

Thus H_0 is an even-dimensional subspace and the spectrum of the operator B_i on this space is symmetrical with respect to zero. Obviously, φ is an eigenvector of the operator $A - \lambda_0 B$

$$A\varphi - \lambda_0 B\varphi = i\omega_0\varphi \qquad (1.7)$$

Theorem 1.1. Let the conditions (1) to (5) hold, λ_0 be a double eigenvalue of the operator T and the operator B_i be nondegenerate on the characteristic subspace H_0 corresponding to λ_0. Then λ_0 is the branch point of a cycle for (1.1).

Proof. It is sufficient to verify that condition (3.6) of Theorem 3.1 of [1] holds. We note that the operator $L = I - \lambda A^{-1}B - i\omega A^{-1} : H_1 \to H_1$ is normal for any real λ and ω. Indeed, simple calculations show that its real L_r and imaginary L_i parts have the form

$$L_r = I - \lambda A^{-1}B_r, \qquad L_i = -\lambda A^{-1}B_i - \omega A^{-1} \qquad (1.8)$$

Condition (4) implies that L_r and L_i commute. The normality of L implies that $\ker L = \ker L^*$. Taking into account the conditions (1) and (2) we can now easily deduce that φ is the only eigenvector of the operator $A - \lambda_0 B$ corresponding to the eigenvalue $i\omega_0$

$$A\varphi - \lambda_0 B^*\varphi = -i\omega_0\varphi \qquad (1.9)$$

Taking into account (1.7) we now obtain

$$\operatorname{Re}(B\varphi, \varphi)_H = \frac{1}{\lambda_0}\|\varphi\|^2_{H_1} > 0 \qquad (1.10)$$

We note that Theorem 1.1 can be easily generalized. It can e.g. be assumed that the operator A has an imaginary component and that the operator B is subordinate to A^α $(0 < \alpha < 1)$. Naturally, in this case the conditions (4) and (5) are appropriately altered.

The fact that Eqs. (1.3) or (1.4) are invariant under a certain group of transformations (e.g. relative to some representation of a group of peripheral rotations in an operator ring) may be the reason for the existence of a double eigenvalue. This occurs in the following example.

Example. Let r and θ be polar coordinates on a plane and Ω a ring, $\Omega = \{(r, \theta) : 0 < r_1 < r < r_2\}$. Consider the parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + \lambda \left(u + \alpha \frac{\partial u}{\partial \theta}\right) + u^3 \tag{1.11}$$

with real parameters λ, $\alpha \neq 0$. We shall seek its solutions satisfying the boundary conditions

$$u\,|_{r=r_1,\, r_2} = 0 \tag{1.12}$$

in a cylinder $-\infty < t < \infty$; $(r, \theta) \in \Omega$.

The system (1.11) and (1.12) can be treated as an ordinary differential equation in a Hilbert space $H = L_2(\Omega)$, and we must assume

$$Au = -\Delta u, \qquad Bu = u + \alpha\, \partial u / \partial\theta, \qquad Ku = u^3 \tag{1.13}$$

The domain of definition of the operator A is a functional subspace of $W_2^{(2)}(\Omega)$, satisfying the condition (1.12), and D_B is the energy space $H_1 = D(A^{1/2})$ of the operator A. We further have $B_r u = u$ and $B_i u = i^{-1} \alpha \partial u / \partial\theta$, and the conditions (1) – (5) can now be easily verified.

Let us consider the eigenvalue problem

$$-\Delta u = \lambda u, \qquad u\,|_{r=r_1,\, r_2} = 0 \tag{1.14}$$

Using the Fourier expansions in θ we confirm, that the eigenfunctions are

$$u_{mn} = \exp im\theta \psi_{mn}(r) \qquad (m = 0, \mp 1, \ldots;\ n = 1, 2, \ldots) \tag{1.15}$$

Here ψ_{mn} denotes the eigenfunction of the Sturm-Liouville problem corresponding to the n-th (in magnitude) eigenvalue λ_{mn}

$$L_m \psi_{mn} \equiv -\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2}\right)\psi_{mn} = \lambda_{mn}\psi_{mn} \tag{1.16}$$

$$\psi_{mn}(r_1) = \psi_{mn}(r_2) = 0$$

Under the boundary conditions (1.16) the differential operator L_m is positive definite in the space L_2 with the weight r on $[r_1, r_2]$. It is well known that all its eigenvalues λ_{mn} are simple. Moreover, L_m increases strictly with m, therefore by the minimax principle λ_{mn} also increases strictly with m.

We shall show that the numbers λ_{mn} $(m = 0, 1, \ldots;\ n = 1, 2, \ldots)$ are all distinct provided that the quantity $\varepsilon = (r_2 - r_1) / r_1$ does not assume values which belong to a certain enumerable set.

Let us select some fixed natural numbers m, n, p, q and verify, for which values of ε the equation $\lambda_{mn} - \lambda_{pq} = 0$ holds. The substitution $r = r_1(1 + \varepsilon x)$ yields (1.16) in the form

$$-\left[\frac{d^2}{dx^2} + \frac{\varepsilon r_1}{1 + \varepsilon x}\frac{d}{dx} - \frac{m^2 r_1^2 \varepsilon^2}{(1 + \varepsilon x)^2}\right]\psi_{mn} = \lambda_{mn} r_1^2 \varepsilon^2 \psi_{mn}, \ \psi_{mn}\Big|_{x=0,1} = 0 \tag{1.17}$$

The perturbation theory implies that the eigenvalue $\lambda_{mn} r_1^2 \varepsilon^2$ is analytic in ε and tends to $n^2\pi^2$ at $\varepsilon \to 0$. From this we deduce that $\lambda_{mn} - \lambda_{pq}$ is analytic in ε and not identically zero when $n \neq q$. But the latter is also true when $n = q$ and $m \neq p$ since $\lambda_{mq} > \lambda_{pq}$ when $m > p$. Thus the set Σ_{mnpq} of those ε for which $\lambda_{mn} = \lambda_{pq}$, is at most enumerable. The set Σ which is a union of all Σ_{mnpq}, is also enumerable. If $\varepsilon \overline{\in} \Sigma$, then λ_{0n} $(n = 1, 2, \ldots)$ denote the simple, and λ_{mn} $(m > 0)$ the double eigenvalues of the operator A. The eigenvalues $\mp m\alpha$ of the operator B_i lie on the characteristic subspace of A, corresponding to λ_{mn} $(m > 0)$, consequently B_i is non-degenerate. Applying Theorem 1.1, we now arrive at the following result.

Let $\varepsilon \overline{\in} \Sigma$. Then a sequence of critical values λ_{mn} $(m, n = 1, 2, \ldots)$ of the parameter λ exists, which are the branch points of the cycle. It can easily be shown that the passage of the parameter λ through the values λ_{0n} $(n = 1, 2, \ldots)$ is accompanied by the branching of the stationary solutions of the system (1.11) and (1.12) from zero.

We note that the auto-oscillating solution of the Navier-Stokes equations investigated in [2, 3], can also be included in the scheme described above.

2. Representation of a cycle in terms of power series. In [1] the Liapunov-Schmidt method was used to study a periodic, auto-oscillating mode of motion of a viscous fluid which occurs when the Reynolds number (or some other parameter γ) passes through its critical value. We also consider a more general problem of appearance of a cycle for an ordinary differential equation in the Banach space X

$$\omega\, dv / d\tau + Av = K(v, \delta) \tag{2.1}$$

where ω is the unknown cyclic frequency, $t = \omega^{-1}\tau$ is time, A is a linear (unbounded) operator and K is a nonlinear operator dependent analytically on the vector $v \in X$ and on the numerical parameter $\delta = \gamma - \gamma_0$ near the point $v = 0,\ \delta = 0$

$$K(v, \delta) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} K_{mn} v^m \delta^n, \qquad K_{1n} = -B_n, \qquad K_{10} = 0 \tag{2.2}$$

We assume that γ_0 denotes the critical value of the parameter γ and that the operator A has a pair of purely imaginary eigenvalues $\mp i\omega_0 \neq 0$. We seek the nontrivial solutions of (2.1), 2π-periodic in τ.

The Navier-Stokes equations represent one particular case of (2.1), another is given by e.g. the equations of motion of a viscoelastic body.

When specific problems are solved, there is usually no need to construct the branching equation, the auto-oscillating mode can be sought directly in the form of a power series. The distinctive feature of the method lies in the fact that the convergence of the series need not be proved separately; the conditions of solvability of equations used to determine the coefficients are identical to those of the theorems of [1]. If formal series (or even their first few terms) satisfying the equations in question can be constructed, they will automatically be found to be convergent. Let us consider the case of Theorem 1.4 of [1]. We assume for definiteness that the auto-oscillations appear when $\delta > 0$. Setting $\delta = \varepsilon^2$, we seek a solution of (2.1) in the form of series (*)

$$v(\tau) = \sum_{k=1}^{\infty} \varepsilon^k v_k(\tau), \qquad \omega = \sum_{k=0}^{\infty} \varepsilon^k \omega_k \tag{2.3}$$

Inserting these series into (2.1) and equating the coefficients of like powers of ε in both parts of the equation, we arrive at the following sequence of equations for the unknown 2π-periodic functions v_k and numbers ω_k:

*) It is already clear that $\omega_{2n+1} = 0$; they were retained in (2.1) for the sake of greater symmetry in the formulas that follow.

$$\omega_0 \frac{dv_1}{d\tau} + Av_1 = 0 \tag{2.4}$$

$$\omega_0 \frac{dv_2}{d\tau} + Av_2 = -\omega_1 \frac{dv_1}{d\tau} + K_{20}v_1^2 \tag{2.5}$$

$$\omega_0 \frac{dv_3}{d\tau} + Av_3 = -\omega_1 \frac{dv_2}{d\tau} - \omega_2 \frac{dv_1}{d\tau} - B_1v_1 + K_{20}{}^\circ(v_1, v_2) + K_{30}(v_1, v_1, v_1)$$

$$K_{20}{}^\circ(v_1, v_2) = K_{20}(v_1, v_2) + K_{20}(v_2, v_1) \tag{2.6}$$

For $p = 2, 3, ..$ we have

$$\omega_0 \frac{dv_p}{d\tau} + Av_p = -\sum_{k=1}^{p-1} \omega_k \frac{dv_{p-k}}{d\tau} -$$

$$-\sum_{2n+m=p} B_n v_m + \sum_{j_1+\ldots+j_m+2n=p} K_{mn}(v_{j_1}, \ldots v_{j_m}) \equiv f_p \tag{2.7}$$

$$(n = 0, 1, \ldots; j_1, j_2, \ldots j_m = 1, 2, \ldots)$$

Let us solve these equations one after the other. It was shown in [1] that by displacing the time $\tau \to \tau + h$ we can obtain the vector function v_k in the following form

$$v_k = u_k + \alpha_k \psi \quad (\alpha_1 > 0) \quad \int_0^{2\pi} (u_k(\tau), \Phi) e^{-i\tau} d\tau = 0 \tag{2.8}$$

$$\psi = \varphi e^{i\tau} + \varphi^* e^{-i\tau}, \qquad A\varphi + i\omega_0\varphi = 0, \qquad A^*\Phi - i\omega_0\Phi = 0$$

$$(\varphi, \Phi) = 1$$

with the unknown constants α_k $(k = 1, 2, \ldots)$.

Equation (2.4) yields

$$v_1(\tau) = \alpha_1 \psi(\tau) \tag{2.9}$$

Condition of solvability of the equation

$$\omega_0 \frac{du}{d\tau} + Au = f \tag{2.10}$$

with 2π-periodic functions f and u has the form

$$\int_0^{2\pi} (f(\tau), \Phi) e^{-i\tau} d\tau = 0 \tag{2.11}$$

with the additional condition

$$\int_0^{2\pi} (u(\tau), \Phi) e^{-i\tau} d\tau = 0 \tag{2.12}$$

We note that the 2π-periodic solution u of (2.10) is unique. The operator R is found by setting $u = Rf$.

Applying (2.12) to Eq. (2.5) we find that $\omega_1 = 0$. The constant α_2 remains unknown, while the vector function u_2 can be found from (2.8) (with $k = 2$) and

$$u_2 = \alpha_1^2 u_{20}, \qquad \omega_0 \frac{du_{20}}{d\tau} + Au_{20} = K_{20}(\psi, \psi) \tag{2.13}$$

We can write the vector function u_{20} in the form

$$u_{20} = z_0 + z_2 e^{2i\tau} + z_2{}^* e^{-2i\tau}$$

$$z_0 = A^{-1} K_{20}{}^\circ (\varphi, \varphi^*), \qquad z_2 = (A + 2i\omega_0 I)^{-1} K_{20} (\varphi, \varphi) \tag{2.14}$$

The condition of solvability of (2.6) has the form

$$-2\pi i \omega_2 \alpha_1 + g_{030} \alpha_1{}^3 + g_{110} \alpha_1 = 0$$

$$g_{030} = \int_0^{2\pi} (K_{030} \psi^3 + \overset{\circ}{K}_{20} (\psi, u_{20}), \Phi) e^{-i\tau} d\tau \tag{2.15}$$

$$g_{110} = -2\pi (B_1 \varphi, \Phi)$$

and this yields α_1 and ω_2

$$\alpha_1 = \left[-\frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{030}} \right]^{1/2}, \qquad 2\pi\omega_2 = \operatorname{Im} g_{110} - \operatorname{Re} g_{110} \frac{\operatorname{Im} g_{030}}{\operatorname{Re} g_{030}} \tag{2.16}$$

If $\operatorname{Re} g_{110} \operatorname{Re} g_{030} < 0$, the cycle branches out for $\delta > 0$. If on the other hand the above quantity is positive, the cycle appears when $\delta < 0$ [1]. The right hand side of (2.6) can be written as

$$f_3 = 2\alpha_1 \alpha_2 K_{20} (\psi, \psi) + q_3$$

$$q_3 = -\omega_2 \alpha_1 \frac{d\psi}{d\tau} - \alpha_1 B_1 \psi + \alpha_1{}^3 [K_{20}{}^\circ (\psi, u_{20}) + K_{30} \psi^3] \tag{2.17}$$

and the vector function q_3 can be assumed known. In accordance with (2.17) and (2.13) we can write the vector function u_3 in the form

$$u_3 = 2\alpha_1 \alpha_2 u_{20} + w_3, \qquad w_3 = R q_3 \tag{2.18}$$

The condition of solvability of (2.7) with $p = 4$ yields ω_3 and α_2. To obtain them we write f_4 in the form

$$f_4 = \alpha_2 f_0 + \alpha_2{}^2 K_{20} \psi^2 + 2\alpha_1 \alpha_3 K_{20} \psi^2 - \omega_3 \alpha_1 \frac{d\psi}{d\tau} + f_{40} \tag{2.19}$$

$$f_0 = -\omega_2 \frac{d\psi}{d\tau} - B_1 \psi + 3\alpha_1{}^2 [K_{20}{}^\circ (u_{20}, \psi) + K_{30} \psi^3] \tag{2.20}$$

$$f_{40} = -\omega_2 \alpha_1{}^2 \frac{du_{20}}{d\tau} - \alpha_1{}^2 B_1 u_{20} + \alpha_1 K_{20}{}^\circ (\psi, w_3) + \alpha_1{}^4 K_{20} (u_{20}, u_{20}) +$$

$$+ \alpha_1{}^2 K_{21} \psi^2 + \alpha_1{}^4 K_{30}{}^\circ (\psi, \psi, u_{20}) + \alpha_1{}^4 K_{40} \psi^4 \tag{2.21}$$

We call the trigonometric polynomial even (odd) if it contains the harmonics of $\exp 2ni\tau$ ($\exp (2n+1)i\tau$) only.

Note that w_3 is an odd trigonometric polynomial and f_{40} is an even one. Therefore the insertion of f_4 into the condition of solvability (2.11), with (2.15) and (2.16) taken into account, yields

$$\alpha_2 \left(-2\pi i \omega_2 + g_{110} - 3g_{030} \frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{030}} \right) - 2\pi i \alpha_1 \omega_3 = 0 \tag{2.22}$$

Since $\operatorname{Re} g_{110} \neq 0$, from (2.22) follows $\alpha_2 = \omega_3 = 0$. Thus we have

$$v(\tau) = \varepsilon \alpha_1 \psi + \varepsilon^2 \alpha^2{}_1 u_{20} + O(\varepsilon^3), \qquad \omega = \omega_0 + \omega_2 \varepsilon^2 + O(\varepsilon^4) \tag{2.23}$$

Let us describe the procedure of computing the consecutive terms of the expansion (2.23). Assume

$$u_1, u_2, \ldots u_{p-2}; \quad \alpha_1, \alpha_2, \ldots \alpha_{p-3}; \quad \omega_1, \omega_2, \ldots \omega_{p-2} \ (p \geqslant 5)$$

to be already known. Then f_{p-1} can be written in the form

$$f_{p-1} = 2\alpha_1\alpha_{p-2}K_{20}\psi^2 + h_{p-1} \tag{2.24}$$

where the vector function h_{p-1} can be assumed known. In accordance with (2.24), we obtain

$$u_{p-1} = 2\alpha_1\alpha_{p-2}u_{20} + w_{p-1}, \qquad w_{p-1} = Rh_{p-1} \tag{2.25}$$

Inserting (2.25) into (2.7), the latter defining f_p, we arrive at

$$f_p = -\omega_{p-1}\alpha_1 d\psi/d\tau + \alpha_{p-2}f_0 + 2\alpha_1\alpha_{p-1}K_{20}\psi^2 + f_{p0} \tag{2.26}$$

containing the known vector function f_{p0}. Inserting f_p into the condition (2.11) of solvability of (2.7), we obtain

$$-2\pi i\alpha_1\omega_{p-1} + \alpha_{p-2}\left(-2\pi i\omega_2 + g_{110} - 3g_{030}\frac{\operatorname{Re} g_{110}}{\operatorname{Re} g_{030}}\right) = r_p$$

$$r_p = \int_0^{2\pi} (f_{p0}, \Phi)\, e^{-i\tau}d\tau \tag{2.27}$$

which readily yields ω_{p-1} and α_{p-2}. The formula (2.25) now gives u_{p-1} The process can be continued in the same manner. We note that the vector function u_p is an even trigonometric polynomial if p is even, and odd if p is odd. This can easily be shown by induction with respect to p. Furthermore, the proof of Theorem 2.2 of [1] implies that $\alpha_{2n} = 0$ $(n = 1, 2, \ldots)$.

On some occasions different power series expansions of the auto-oscillations are found useful. Let us seek a solution of (2.1) in the form

$$v(\tau) = \alpha\psi(\tau) + u(\tau), \qquad \int_0^{2\pi} (u(\tau), \Phi)\, e^{-i\tau}d\tau = 0 \tag{2.28}$$

and the vector function u as well as the numerical parameters ω and δ in the form of series in powers of the amplitude α

$$\omega = \omega_0 + \sum_{k=1}^{\infty} \mu_k\alpha^{2k}, \qquad \delta = \sum_{k=1}^{\infty} \delta_k\alpha^{2k}, \qquad u = \sum_{k=2}^{\infty} \alpha^k w_k \tag{2.29}$$

Expansions (2.29) are more general than (2.3). They remain valid under the conditions of Theorems 2.1 and 3.1 of [1] (see the proof of Theorem 2.1) and are of particular interest especially in the case when the equation contains a certain complementary parameter η and, when at certain values of this parameter, the radicand in (2.16) changes its sign and the expansions (2.3) no longer hold. We note that similar amplitude expansions were proposed by Landau [4] ([5] deals with the proof of the Landau method).

Inserting (2.28) and (2.29) into (2.1), we obtain

$$v(\tau) = \alpha\psi + \alpha^2 u_{20} + O(\alpha^3)$$

$$\delta = -\frac{\operatorname{Re} g_{030}}{\operatorname{Re} g_{110}}\alpha^2 + O(\alpha^4) \tag{2.30}$$

$$\omega = \omega_0 + \mu_1\alpha^2 + O(\alpha^4), \qquad \mu_1 = \operatorname{Im} g_{030} - \operatorname{Re} g_{030}\frac{\operatorname{Im} g_{110}}{\operatorname{Re} g_{110}}$$

3. Stability of the fundamental mode and the auto-oscillation.

We can use the perturbation method to investigate the stability of the branched-out auto-oscillating mode for the values of the parameter γ close to the critical one. In fact, the corresponding linearized equation contains a small parameter $\delta = \gamma - \gamma_0$ (or its fractional power). When $\delta = 0$, this equation becomes a linearized equation corresponding to the fundamental stationary mode with $\gamma = \gamma_0$. If this mode shows gross instability (the stability spectrum containing the points of the right semiplane), it remains such when δ are small (and so does the branched-out cycle). For this reason we only need to consider the case when all the points of the stability spectrum of the fundamental mode with the exception of $in\,\omega_0$ $(n = 0, \mp 1, \ldots)$ lie, for $\gamma = \gamma_0$, within the left semiplane. In the end we find that it suffices to learn in which direction the excepted points are displaced as a result of a perturbation. In fact, it is enough to consider that eigenvalue σ, which becomes $\sigma_0 = 0$ when $\delta = 0$. The remaining eigenvalues have the form $\sigma + in\omega_0$. The eigenvalue $\sigma_0 = 0$ is double, and it usually splits under the action of a perturbation into two simple ones. One of them is necessarily zero and is always found in the stability spectrum of the auto-oscillation. We therefore turn our attention to the value of σ different from zero. We find that if $\mathrm{Re}\,\sigma < 0$, the cycle is stable, if $\mathrm{Re}\,\sigma > 0$ the cycle is unstable.

We must remember that the perturbation theory can describe, generally speaking, only the behavior of compact fragments of the spectrum, and not of the whole spectrum. For this reason an *a priori* estimate is needed, uniform over small δ of the real parts of those points of the stability spectrum which lie in the right semiplane. Otherwise the assertion that no eigenvalues appear in the right semiplane at a distance from the imaginary axis would not be possible to be made. Of course, instability can be proved without an *a priori* estimate, but the latter is necessary if stability is to be shown conclusively. Such difficulties are however not encountered in the problems which are dealt with in the present paper; the monodromy operator is completely continuous and analytically depends on ε, and only a finite number of multiplicators can be found outside the unit circle. Justification for the use of the linearizing procedure in the problem on stability of periodic motions of a fluid is given in [6, 7].

We restrict ourselves to the case of the Theorems 2.1 and 3.2 of [1]. The method however can also be used under the conditions of the remaining theorems of [1]. We assume for definiteness that the auto-oscillations appear when $\delta > 0$.

Let us linearize (2.1) in the neighborhood of the periodic solution $v(\tau)$. Seeking the solutions of the form $e^{\sigma\tau}w(\tau)$, where $w(\tau)$ is a 2π-periodic vector function, we arrive at the following spectral problem

$$\omega \frac{dw}{d\tau} + \sigma w + Aw = K_v{}'(v, \delta)\, w \tag{3.1}$$

Its right-hand side contains the Frechet derivative with respect to v of the operator K. Using the expansions (2.3), we can reduce (3.1) to the form

$$\omega_0 \frac{dw}{d\tau} + \sigma w + Aw + \sum_{k=2}^{\infty} \varepsilon^k \omega_k \frac{dw}{d\tau} = \sum_{k=1}^{\infty} \varepsilon^k R_k(\tau)\, w \tag{3.2}$$

where $R_1, R_2, \cdots$ are linear operators. In particular, we have

$$R_1 w = \overset{\circ}{K}_{20}(v_1, w), \quad R_2 w = \overset{\circ}{K}_{20}(v_2, w) + \overset{\circ}{K}_{30}(v_1, v_1, w) - B_1 w \tag{3.3}$$

Equations (3.1) or (3.2) are obviously satisfied if we set $\sigma = 0$ and $w = dv / d\tau$. We seek the nonzero eigenvalue σ vanishing when $\varepsilon \to 0$, and the corresponding eigenvector w in the following form:

$$\sigma = \sum_{k=1}^{\infty} \varepsilon^k \sigma_k, \quad w = \sum_{k=0}^{\infty} \varepsilon^k w_k, \quad w_0(\tau) = \varphi e^{i\tau} \tag{3.4}$$

The validity of the expansions (3.4) follows from the perturbation theory. Inserting (3.4) into (3.2) and equating the coefficients of like powers of ε in both sides of the equation, we obtain

$$\omega_0 \frac{dw_1}{d\tau} + Aw_1 = -\sigma_1 w_0 + K_{20}^{\circ}(v_1, w_0) \tag{3.5}$$

$$\omega_0 \frac{dw_2}{d\tau} + Aw_2 = -\sigma_1 w_1 - \sigma_2 w_2 - \omega_2 \frac{dw_0}{d\tau} + R_1 w_1 + R_2 w_0 \tag{3.6}$$

and analogous expressions for w_3, w_4, ... The requirement that the right-hand side of (3.5) satisfies the condition of solvability (2.11) yields, with (3.4) and (2.9) taken into account,

$$\sigma_1 = 0, \qquad w_1 = \alpha_1 z_0 + 2\alpha_1 z e^{2i\tau} \tag{3.7}$$

where the constant α_1 is given by (2.16) and the vectors z_0 and z by [1]

$$z_0 = A^{-1} K_{20}^{\circ} (\varphi, \varphi^*), \qquad z = (A + 2i\omega_0 I)^{-1} K_{20} (\varphi, \varphi) \tag{3.8}$$

Applying the condition of solvability (2.11) to (3.6) we obtain (3.9)

$$\sigma_2 = -i\omega_2 - (B_1\varphi, \Phi) + 2\alpha_1^2 (K_{20} (\varphi, z_0) + K_{20}^{\circ} (\varphi^*, z) + K_{30}^{\vee} (\varphi, \varphi, \varphi^*), \Phi)$$

and, taking (2.15) and (2.16) into account, we have

$$\operatorname{Re} \sigma_2 = -{}^1/_2\pi \operatorname{Re} g_{110} = \operatorname{Re} (B_1\varphi, \Phi) \tag{3.10}$$

The vector function w_2 can be found from (3.6), and the remaining w_k and σ_k can be obtained in a similar manner. Stability of the fundamental mode can be established by just considering (3.1) with $v = 0$. Repeating the procedure given above, we obtain the following expression for the eigenvalue σ :

$$\sigma = \sigma_2 \delta + O(\delta^2), \qquad \operatorname{Re} \sigma_2 = -\operatorname{Re} (B_1\varphi, \Phi) \tag{3.11}$$

and consequently arrive at the following theorem.

Theorem 3.1. Let the conditions of Theorem 3.1 (or 2.1) of [1] hold. Then the fundamental mode is asymptotically stable for small δ if $\delta \operatorname{Re} (B_1\varphi, \Phi) > 0$, and unstable if $\delta \operatorname{Re} (B_1\varphi, \Phi) < 0$. If the conditions of Theorem 3.2 (or 2.2) of [1] also hold, then the branching-out cycle is stable when $\operatorname{Re} (B_1\varphi, \Phi) < 0$, and unstable when $\operatorname{Re} (B_1\varphi, \Phi) > 0$.

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